TRINITY COLLEGE

IN NOMINE DOMINI

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	95	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed 65% (98 Marks)
This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(6 marks)**

Suppose the cost function, C, for producing x units of a product is given by:

$$C(x) = 6000 + 88x - 0.6x^2 + 0.002x^3$$

(a) Determine the cost of producing the 150th unit. (3 marks)

Solution
$C'(x) = 88 - 1.2x + 0.006x^2$
$C(149) = 88 - 1.2(149) + 0.006(149)^2$
$C'(149) = 42.41$
Specific behaviours
✓ determines $C'(x)$ ✓ substitutes 149 into $C'(x)$ ✓ determines $C'(149) = \$42.41$

ALT

$C(150) - C(149)$

$12450 - 12407.30$

$\$42.70$

✓ calc $C(150)$
✓ calc $C(149)$
✓ calc diff.

(b) Determine, with justification, the value of x for which the marginal cost $C'(x)$ is a minimum. (3 marks)

Solution
$C''(x) = -1.2 + 0.012x$
$0 = -1.2x + 0.012x$
$x = 100$
Specific behaviours
✓ determines $C''(x)$ ✓ indicates that $C''(x) = 0$ ✓ solves for $x = 100$

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Question 9 **(8 marks)**

Due to the natural variability in the size, density and so on of fruit and syrup used in canning peaches, a cannery purposely overfills the cans it produces. However, some cans still end up being underweight and the probability p that a canning machine produces such a can is known to be constant. Random samples of n cans are taken from the machine and the number X of underweight cans in each sample is recorded. The mean and standard deviation of X are 2.16 and 1.44 respectively.

(a) X is a discrete random variable. Explain why it is **discrete** and **random**. (2 marks)

Solution
Discrete: it can only take integer values between 0 and n (or a countable number of distinct values). Random: the number of underweight cans in a sample is not predictable (or its possible values are numerical outcomes resulting from a random process).
Specific behaviours
✓ explains discrete ✓ explains random (Do not accept 'because sampling is random', etc)

(b) Name the distribution of X and determine the value of n and the value of p. (3 marks)

Solution
Distribution of X is binomial. $np = 2.16$ $np(1-p) = 1.44^2$ Solving simultaneously gives $n = 54$ and $p = \frac{1}{25} = 0.04$.
Specific behaviours
✓ states binomial distribution ✓ forms simultaneous equations ✓ value of p and value of n

(c) Another canning machine produces an underweight can with a probability of 0.015. Determine the probability that when a random sample of 30 cans from this machine are weighed

(i) exactly one of the cans is underweight. (2 marks)

Solution
Let be the number of underweight cans, so that $Y \sim B(30, 0.015)$. $P(Y = 1) = 0.2903$.
Specific behaviours
✓ states distribution ✓ correct probability

(ii) more than one of the cans are underweight. (1 mark)

Solution
$P(Y \geq 2) = 0.0742$
Specific behaviours
✓ correct probability

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Question 10 **(8 marks)**

A small body moving in a straight line has an initial velocity of 15 cm/s as it leaves point P. The acceleration of the body at time t seconds is $6 - 1.5t$ cm^2/s, $t \geq 0$.

(a) Determine the displacement of the body relative to P after 2 seconds. (4 marks)

Solution
$v = \int 6 - 1.5t\,dt = 6t - 0.75t^2 + c$ $t = 0, v = 15 \Rightarrow c = 15$ $v(t) = 6t - 0.75t^2 + 15$ $x(2) = \int_0^2 v(t)\,dt \quad \text{OR} \quad x(t) = 3t^2 - 0.25t^3 + 15t$ $= 40 \text{ cm}$
Specific behaviours
✓ antidifferentiates acceleration, with constant   ✓ obtains expression for velocity   ✓ integral for change in displacement OR displacement function   ✓ correct displacement

(b) Determine the maximum velocity of the body. (2 marks)

Solution
$a = 0 \Rightarrow t = 4$ $v(4) = 27 \text{ cm/s}$
Specific behaviours
✓ indicates time   ✓ correct maximum velocity

(c) Determine the maximum displacement of the body relative to P. (2 marks)

Solution
$v = 0 \Rightarrow t = 10$ $x(10) = \int_0^{10} v(t)\,dt = 200 \text{ cm}$
Specific behaviours
✓ indicates time   ✓ correct maximum displacement

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Question 11 **(8 marks)**

The following table shows the probability distribution of a discrete random variable X, where k is a constant.

x	-2	0	1	3
$P(X = x)$	$4k^2$	0.15	$2k$	0.1

(a) Determine the value of k. (3 marks)

Solution
$4k^2 + 0.15 + 2k + 0.1 = 1$ $4k^2 + 2k - 0.75 = 0$ $k = \frac{1}{4} = 0.25 \ (k \geq 0)$
Specific behaviours
✓ indicates sum of probabilities is 1 ✓ forms equation ✓ solves and states single value of k

(b) Determine $E(X)$. (2 marks)

Solution
$E(X) = (-2)(0.25) + (0)(0.15) + (1)(0.5) + (3)(0.1)$ $= \frac{3}{10} = 0.3$
Specific behaviours
✓ indicates correct method ✓ correct expected value

(c) Given that $\text{Var}(X) = 2.31$, determine the following for the discrete random variable Z:

(i) $E(Z)$ when $Z = 5X - 3$. (1 mark)

Solution
$E(Z) = 5E(X) - 3 = 5(0.3) - 3 = -\frac{3}{2} = -1.5$
Specific behaviours
✓ correct value

(ii) $\text{Var}(Z)$ when $Z = \frac{X}{3} + 2$. (1 mark)

Solution
$\text{Var}(Z) = \left(\frac{1}{3}\right)^2 \text{Var}(X) = \frac{1}{9}(2.31) = \frac{77}{300} = 0.25\overline{6}$
Specific behaviours
✓ correct value

(iii) The standard deviation of Z when $Z = 5(2 - X)$. (1 mark)

Solution
$\sigma_Z = 5\sqrt{\text{Var}(X)} = 5\left(\frac{\sqrt{231}}{10}\right) = \frac{\sqrt{231}}{2} \approx 7.6$
Specific behaviours
✓ correct value

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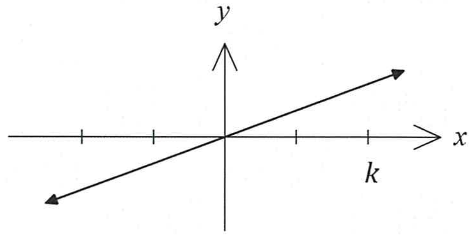
Question 12 **(7 marks)**

(a) Consider the function $f(x) = mx$, where m is a constant. The graph of $y = f(x)$ is shown at right, k is a constant and

$$\int_0^k f(x)\,dx = 4.$$



Determine the value of

(i) $\int_{-k}^{k} f(x)\,dx.$ (1 mark)

Solution
$-4+4=0$
Specific behaviours
✓ correct value

(ii) $\int_{-k}^{0} 2f(x+k)\,dx.$ (2 marks)

Solution
$\int_{-k}^{0} 2f(x+k)\,dx = 2\int_0^k f(x)\,dx = 2(4) = 8$
Specific behaviours
✓ uses linearity to move constant outside integral ✓ correct value

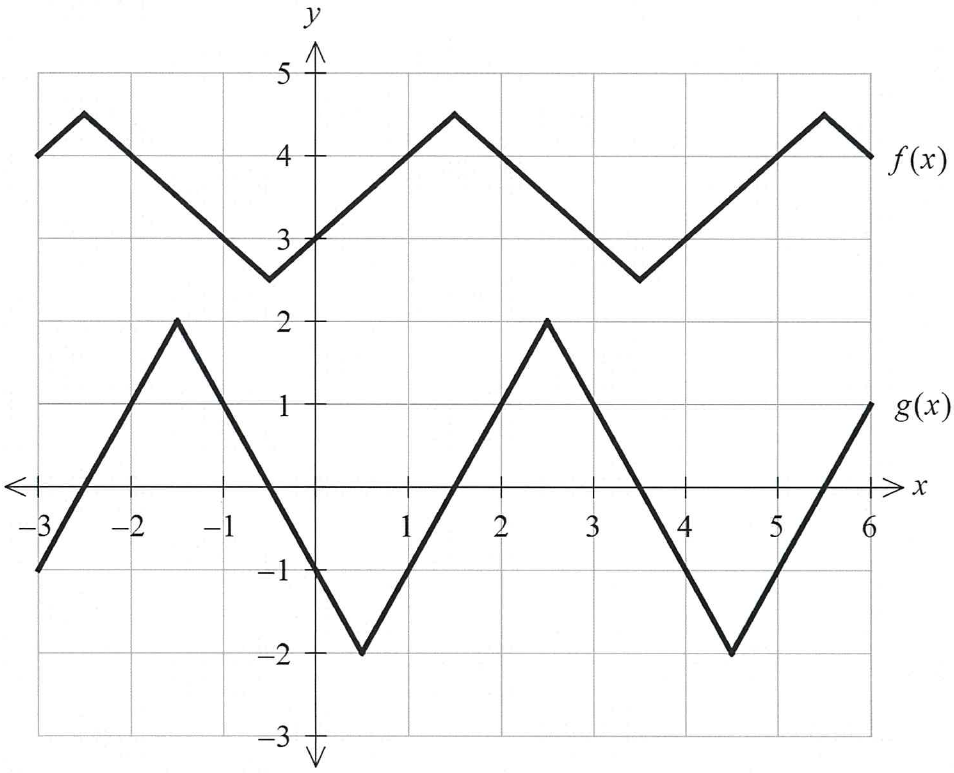
(b) The polynomial function $g(x)$ is such that $\int_{-2}^{5} g(x)\,dx = 10.$

Determine the value of $\int_{-2}^{2} (2 - g(x))\,dx + \int_{2}^{5} (2x - g(x))\,dx.$ (4 marks)

Solution
$I = \int_{-2}^{2} (2x - g(x))\,dx + \int_{2}^{5} (2 - g(x))\,dx$ $= \int_{-2}^{2} (2)\,dx - \int_{-2}^{2} (g(x))\,dx + \int_{2}^{5} (2x)\,dx - \int_{2}^{5} (g(x))\,dx$ $= [2x]_{-2}^{2} - \int_{-2}^{5} (g(x))\,dx + [x^2]_2^5$ $= (4+4) - 10 + (25-4)$ $= 19$
Specific behaviours
✓ uses linearity to obtain four integrals ✓ uses additivity to combine integrals of $g(x)$ ✓ evaluates $2x$ integral correctly ✓ correct value

Question 13 **(8 marks)**

The graphs of the continuous functions $y = f(x)$ and $y = g(x)$ are shown below.



(a) Evaluate the derivative of $f(x)g(x)$ at $x = -2$. (2 marks)

Solution
$$\frac{d}{dx}\big(f(x)g(x)\big)_{x=-2} = f'(-2)g(-2) + f(-2)g'(-2)$$ $$= (-1)(1) + (4)(2)$$ $$= 7$$
Specific behaviours
✓ indicates correct values of $f'(-2)$ and $g'(-2)$ ✓ correctly evaluates derivative

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(b) Evaluate the derivative of $f(g(x))$ at $x = 5$. (3 marks)

Solution
$$\frac{d}{dx}f(g(x))_{x=5} = f'(g(5)) \times g'(5)$$ $$= f'(-1) \times g'(5)$$ $$= -1 \times 2$$ $$= -2$$
Specific behaviours
✓ indicates correct application of chain rule ✓ indicates correct value of $f'(g(5))$ ✓ correctly evaluates derivative

(c) Evaluate the derivative of $\dfrac{g'(x)}{f(x)}$ at $x = 0$. (3 marks)

Solution
$$\frac{d}{dx}\left(\frac{g'(x)}{f(x)}\right)_{x=0} = \frac{g''(0)f(0) - g'(0)f'(0)}{(f(0))^2}$$ $$= \frac{(0)(3) - (-2)(1)}{3^2}$$ $$= {}^{2}/_{9}$$
Specific behaviours
✓ indicates correct application of quotient rule ✓ indicates correct value of $g''(0)$ ✓ correctly evaluates derivative

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Question 14 **(9 marks)**

A full water tank takes 31 seconds to empty. The volume V litres of water in the tank, t seconds after emptying began, is changing at a rate given by

$$\frac{dV}{dt} = \sqrt[3]{4t+1} - 5, \qquad 0 \le t \le 31.$$

(a) Determine the initial rate of change of volume. (1 mark)

Solution
$\frac{dV}{dt} = \sqrt[3]{4(0)+1} - 5 = -4 \text{ L/s}$
Specific behaviours
✓ correct rate of change

(b) Use the increments formula to estimate the volume of water that empties from the tank during the first one-tenth of a second. (2 marks)

Solution
$\delta V \approx \frac{dV}{dt}\delta t$ $\approx -4 \times {}^{1}/_{10} \approx -0.4$ An estimated 0.4 L empties from the tank.
Specific behaviours
✓ shows use of the increments formula   ✓ correct estimate

(c) Determine the initial volume of water in the tank. (3 marks)

Solution
$\Delta V = \int_0^{31} \sqrt[3]{4t+1} - 5\, dt$ $= -38$ Hence tank initially contained 38 L.
Specific behaviours
✓ writes correct integral   ✓ evaluates total change   ✓ states correct initial volume

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(d) Determine the time, to the nearest 0.01 second, when the tank is half full. (3 marks)

Solution
$V = \int \sqrt[3]{4t+1} - 5\,dt = \frac{3}{16}(4t+1)^{\frac{4}{3}} - 5t + c$ But when $t = 0, V = 38$ and so $c = {}^{605}/_{16} = 37.8125$ $\frac{3}{16}(4t+1)^{\frac{4}{3}} - 5t + \frac{605}{16} = \frac{38}{2}$ $t = 7.27$ s
Specific behaviours
✓ obtains antiderivative ✓ evaluates constant of integration ✓ solves for time

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Question 15 **(8 marks)**

A bag contains four red and three blue balls. Two balls are drawn at random and in succession from the bag. At each draw, if the ball is red it is replaced in the bag, and otherwise the ball is not replaced. Let X be the number of red balls drawn.

(a) Determine $P(X = 2)$. (1 mark)

Solution
$P(X = 2) = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$
Specific behaviours
✓ correct probability

(b) Use exact values to complete the probability distribution table for X below. (3 marks)

x	$P(X = x)$
0	$\frac{1}{7}$
1	$\frac{26}{49}$
2	$\frac{16}{49}$

Solution
$P(X = 0) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$ $P(X = 1) = 1 - \frac{16}{49} - \frac{1}{7} = \frac{26}{49}$
Specific behaviours
✓ calculates $P(X = 0)$ ✓ calculates $P(X = 1)$ ✓ completes table using exact values

(c) Determine the mean and variance of X. (4 marks)

Solution

$$E(X) = 0 \times \frac{1}{7} + 1 \times \frac{26}{49} + 2 \times \frac{16}{49}$$
$$= \frac{58}{49} \quad (\approx 1.184)$$

$$\text{Var}(X) = \left(0 - \frac{58}{49}\right)^2 \times \frac{1}{7} + \left(1 - \frac{58}{49}\right)^2 \times \frac{26}{49} + \left(2 - \frac{58}{49}\right)^2 \times \frac{16}{49}$$
$$= \frac{1046}{2401} \quad (\approx 0.436)$$

Specific behaviours

✓ expression for mean
✓ correct mean
✓ expression for variance
✓ correct variance

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Question 16 **(7 marks)**

The concentration of a drug in the plasma of a monkey, C micrograms per litre, t hours after being administered, can be modelled by $C = C_0e^{kt}$, where C_0 and k are constants. Each dose of the drug increases the existing concentration by $430\ \mu g/L$, and the concentration of the drug is known to halve every 2 hours and 40 minutes.

(a) Show that $k = -0.26$. (1 mark)

Solution
$0.5 = e^{2.\bar{6}k} \rightarrow k = -0.260$
Specific behaviours
✓ correctly forms equation for k using half life

A monkey, with no existing trace of the drug, was administered a first dose at $8{:}05$ am.

(b) Use the model to determine the rate of change of concentration of the drug in the monkey's plasma later that morning at $10{:}45$ am. (2 marks)

Solution
$\frac{dC}{dt} = kC_0e^{kt} = kC$ At $10{:}45$ pm, $t = 2\text{h}\ 40\text{m}$ and so $C = 430 \div 2 = 215$. $\frac{dC}{dt} = -0.26(215)$ $= -55.9\ \mu g/L/h$
Specific behaviours
✓ indicates expression for rate of change   ✓ correctly calculates rate of change

An additional dose is administered every time the concentration falls to $130\ \mu g/L$.

(c) Determine the expected time of day, to the nearest minute, that the third dose will be administered to the monkey. (4 marks)

Solution
Time until second dose is given: $430e^{-0.26t} = 130 \rightarrow t = 4.602$.   New $C_0 = 130 + 430 = 560 \rightarrow C = 560e^{-0.26t}$.   Time from second to third dose: $560e^{-0.26t} = 130 \rightarrow t = 5.618$.   Total time: $T = 4.602 + 5.618 = 10.22 = 10\text{h}\ 13\text{m}$. Hence third dose will be given at $8{:}05 + 10{:}13 = 6{:}18$ pm.
Specific behaviours
✓ time until second dose administered   ✓ indicates new equation for concentration   ✓ time between second and third doses   ✓ correct time of day

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Question 17 **(10 marks)**

A machine learning model is being developed to recognise a pathogen in medical images. The performance of the model is stable and the results of the last 250 runs of the machine are shown in the table below.

		Model recognises a pathogen in image	
		Yes	No
Image contains a pathogen	Yes	154	21
	No	6	69

(a) Determine the probability that the model recognises a pathogen in a randomly selected image that contains a pathogen. (1 mark)

Solution
$p = \dfrac{154}{154 + 21} = \dfrac{154}{175} = \dfrac{22}{25} = 0.88$
Specific behaviours
✓ correct probability

(b) The model is used to check 9 randomly selected images. Determine the probability that it returns exactly one incorrect result. (3 marks)

Solution

Probability of an incorrect result, p:

$$p = \frac{6 + 21}{250} = \frac{27}{250} = 0.108$$

Let X be number of incorrect results, then $X \sim B(9, 0.108)$.

$$P(X = 1) = 0.3896$$

Specific behaviours

✓ calculates probability of incorrect result
✓ states distribution is binomial, with parameters
✓ calculates probability

(c) The model is used to check 30 randomly selected images. Determine the probability that it returns at least 26 correct results. (3 marks)

Solution

Probability of a correct result, p:

$$p = 1 - \frac{27}{250} = \frac{223}{250} = 0.892$$

Let X be number of correct results, then $X \sim B(30, 0.892)$.

$$P(X \geq 26) = 0.7817$$

Specific behaviours

✓ calculates probability of correct result
✓ states distribution is binomial, with parameters
✓ calculates probability

(d) The model is repeatedly used to check batches of 50 randomly selected images that do not contain a pathogen. Determine the mean and standard deviation of the probability distribution for the number of correct results the model produces. (3 marks)

Solution

$$p = \frac{69}{6+69} = \frac{69}{75} = \frac{23}{25} = 0.92$$

$$\mu = np = 50 \times \frac{23}{25} = 46$$

$$\sigma = \sqrt{np(1-p)}$$
$$= \sqrt{46 \times \frac{2}{25}}$$
$$= \frac{2\sqrt{23}}{5} \approx 1.918$$

Specific behaviours

✓ calculates conditional probability
✓ calculates mean
✓ calculates standard deviation

Question 18 **(10 marks)**

A small body moves in a straight line with velocity v cm/s at time t s given by

$$v(t) = 14 + 5\sin\left(\frac{\pi t}{6}\right) - 8\sin\left(\frac{\pi t}{3}\right), \qquad t \geq 0.$$

(a) By viewing the graph of the velocity function on your calculator, or otherwise, state the minimum velocity of the body for $t \geq 0$ to the nearest 0.01 cm/s, and hence explain why the distance travelled by the body in any interval of time will always be the same as the change in displacement of the body. (2 marks)

Solution
$v_{MIN} = 2.29$ cm/s Distance travelled same as change in displacement as the velocity is always positive.
Specific behaviours
✓ states minimum velocity ✓ explanation

(b) Determine the distance travelled by the body between $t = 0$ and $t = 12$. (2 marks)

Solution
$d = \int_0^{12} v(t)\,dt$ $= 168$ cm
Specific behaviours
✓ writes correct integral ✓ correct distance

The distance travelled (x cm) by the body in any 6 second interval from $t = T$ to $t = T + 6$ is given by the function $x(T) = a + b\cos\left(\frac{\pi T}{6}\right)$.

(c) Determine the value of the constant a and the value of the constant b. (2 marks)

Solution
$x(T) = \int_T^{T+6} v(t)\,dt$ $= 84 + \frac{60}{\pi}\cos\left(\frac{\pi T}{6}\right)$ Hence $a = 84$ and $b = \frac{60}{\pi}$.
Specific behaviours
✓ writes integral ✓ uses result to state both values

(d) During the first 20 seconds, there is an 6 second interval in which the distance travelled by the body is a minimum. Using calculus methods, determine when this interval occurs and justify that the distance is a minimum. (4 marks)

Solution
$x'(T) = -10\sin\left(\frac{\pi T}{6}\right)$ $x'(T) = 0$ when $T = 0, 6, 12$ $x''(T) = -\frac{5\pi}{3}\cos\left(\frac{\pi T}{6}\right)$ $x''(0) = -\frac{5\pi}{3}, \quad x''(6) = \frac{5\pi}{3}$ Hence when the interval starts at $T = 6$ seconds, the distance is a minimum since at this time the first derivative of the distance function is zero and the second derivative is positive.
Specific behaviours
✓ obtains derivative and equates to zero ✓ indicates times when derivative is zero ✓ uses second derivative to identify first minimum ✓ states correct time, with justification

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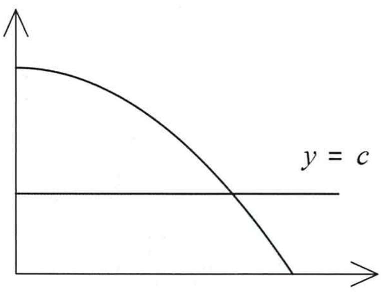
Question 19 **(7 marks)**

The line $y = c$ divides the area in the first quadrant under the curve $y = 16 - x^2$ into two equal halves, as shown in the diagram.

Determine, with reasoning, the value of c.



Solution
Let the curve and line intersect when $x = a$, so that $c = 16 - a^2$. Area above line is area between curve and line: $$A_A = \int_0^a (16 - x^2) - (16 - a^2)\,dx = \frac{2a^3}{3}$$ Area below line is rectangle plus area under curve: $$A_B = a(16 - a^2) + \int_a^4 (16 - x^2)\,dx = 16a - a^3 + \frac{a^3}{3} - 16a + \frac{128}{3} = \frac{128}{3} - \frac{2a^3}{3}$$ Require $A_A = A_B$ and so $$\frac{2a^3}{3} = \frac{128}{3} - \frac{2a^3}{3}$$ $$a = 2\sqrt[3]{4}$$ Hence $c = 16 - \left(2\sqrt[3]{4}\right)^2 = 16 - 8\sqrt[3]{2} \approx 5.921$.
Specific behaviours
✓ expresses c in terms of x-coordinate of intersection ✓ writes integral for upper area ✓ evaluates and simplifies integral ✓ writes expression for lower area ✓ evaluates and simplifies expression ✓ equates expressions and solves for a ✓ substitutes to obtain c

End of questions

Supplementary page

Question number: ____________________

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